A Procedural Model of Lottery Complexity

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- EU representation is nonetheless violated, more so in complex choices.
- If people agree with axioms, why do they violate them?
- Maybe it's not the axioms, but the complexity of applying them.
- We all know the rules of arithmetic... but we all fail at arithmetic.

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- If we think axioms are true but violated due to the complexity in applying them then we need to go beyond the axiomatic framework:
  - In an axiomatic framework: Non-EU behavior $\Leftrightarrow$ Axiom Violations.
  - No way of specifying when a choice/object is complex.

Introduction

- I offer a procedural framework in which:
  - DM uses rules to simplify lotteries before comparing them.
  - The ways they are able/unable to use the rules imply their complexity measures.

Results Preview

- Today:
  - EU $\Leftrightarrow$ No restriction on use of rules.
  - Types of restrictions $\Leftrightarrow$ Measures of complexity aversion.
    - I characterize procedurally the support size cost (Puri 2020) and entropy cost (Mononen 2021).
  - Partition Complexity: a new notion of complexity arising from procedural motivations.

Framework

- Monetary outcomes: $x \in \mathcal{X} = \mathbb{R}$
- Simple lotteries: $\ell \in \mathcal{L} = \Delta(\mathcal{X})$
- Notation:
  1. Degenerate lotteries: δ_x is the lottery yielding x with probability 1.
  2. Mixture: $\ell^* = \ell_1 \alpha \ell_2$ means $\ell^* = \alpha \ell_1 + (1-\alpha)\ell_2$.
  3. Exclusive mixture: $\ell^* = \ell_1 \overline{\alpha} \ell_2$ means $\ell^* = \alpha \ell_1 + (1-\alpha)\ell_2$ and $supp(\ell_1) \cap supp(\ell_2) = \emptyset$.

Model of Comparison

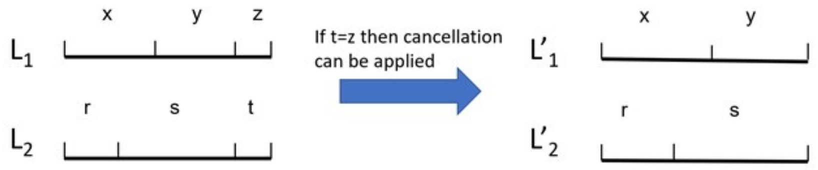
- DM uses rules to make choices.
- DM is modelled via two components:
  - **Rules**: Used to simplify a lottery or to simplify a comparison of two lotteries.
  - **Choice**:
    - Basic choices between simplest lotteries.
    - Complex choices which are deduced by "legal" sequences of application of rules.

Rules

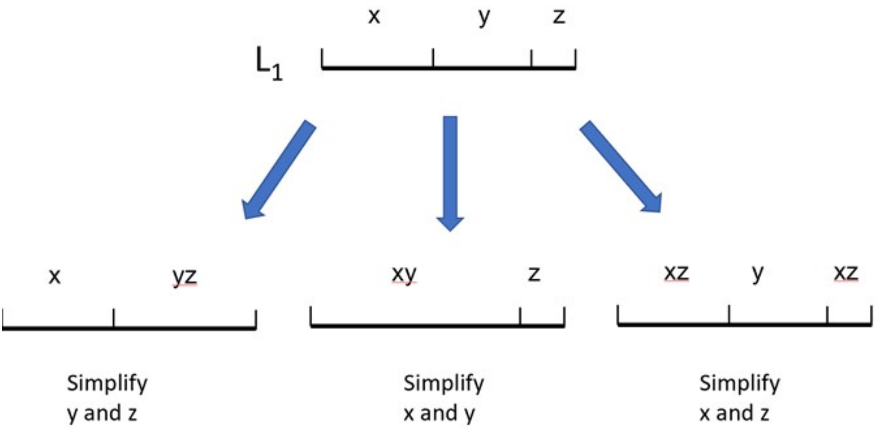
- Rules are asymmetric binary relations on lottery or pairs of lotteries.

Cancellation
$(\ell_1, \ell_2) C (\ell'_1, \ell'_2)$ if $\exists \alpha, x$ such that $(\ell_1, \ell_2) = (\ell'_1 \overline{\alpha} x, \ell'_2 \overline{\alpha} x)$.

- Example:



Rules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Simplification
$\ell_1 S \ell_2$ if $\exists \ell^*, x\alpha y \sim^* z$ such that $\ell_1 = \ell^* \overline{\beta}(x\alpha y)$ and $\ell_2 = \ell^* \beta z$.

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- Simplification lets the DM merge two outcomes into a single one.
- I assume each lottery simplifies to a unique certainty equivalent. In particular the operation is order independent.

Deductions

- The DM uses rules to deduce choice:
  - Denote $(\ell_1, \ell_2) \to (\ell_1', \ell_2')$ if there is a sequence of application of rules which transforms (ℓ_1, ℓ_2) to (ℓ_1', ℓ_2').
  - Least restrictive deduction: $\ell_1 \succeq \ell_2$ if $(\ell_1, \ell_2) \to (\ell_1', \ell_2')$ and $\ell_1' \succeq \ell_2'$.

Model of Comparison: Example

Example of a Comparison Model

1. Rules:
   - Cancellation.
   - Simplification.
2. Choice:
   - $\delta_x \succ \delta_y$ whenever $x > y$.
   - $\ell_1 \succeq \ell_2$ if $(\ell_1, \ell_2) \rightarrow (\ell_1', \ell_2')$ and $\ell_1' \succeq \ell_2'$.

Expected Utility Representation

- Call the previous example the EU-Comparison Model.

Theorem 1

The following are equivalent:

- $\succeq$ arises out of an EU-comparison model.
- $\succeq$ has an expected utility representation with continuous and strictly increasing u.

Some More Rules

Adjacent Simplification

Let ℓ be a lottery with outcomes $x_1 > .. > x_n$, a simplification of ℓ is *adjacent* if it simplifies outcomes x_i and x_{i+1}.

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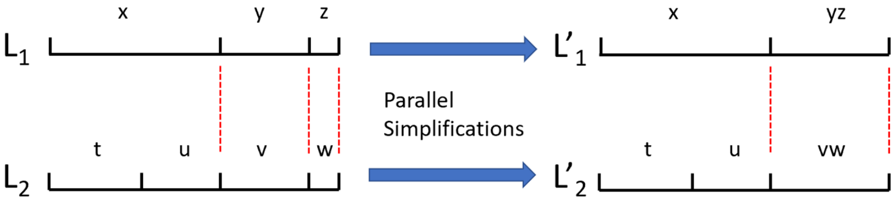
Replacement

$(\ell_1, \ell_2) R(\ell'_1, \ell'_2)$ if $\exists \alpha, x, y,\ \ell'_1 = \ell^*_1 \overline{\alpha} x, \ell_2 = \ell^*_2 \overline{\alpha} x$ and $\ell'_1 = \ell^*_1 \overline{\alpha} y$, $\ell'_2 = \ell^*_2 \overline{\alpha} y$.

Parallel Simplifications



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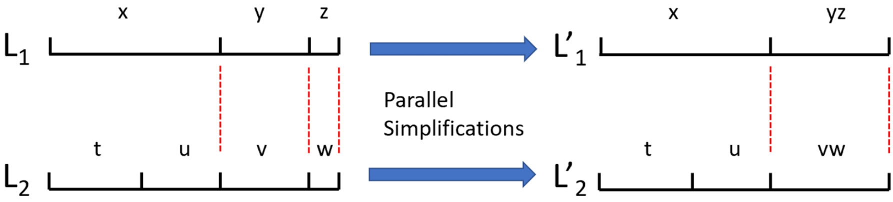

Parallel Simplifications

We say (ℓ_1, ℓ_2) to (ℓ_1', ℓ_2') is a pair of *parallel* simplifications if $\exists \alpha, \beta$ such that $\ell_i = \ell_i^* \overline{\alpha}(x_i \beta y_i)$ and $\ell_i' = \ell_i^* \alpha z_i$.

- A pair of simplifications is parallel if it simplified the same probabilities.

Complexity Aversion

- In the EU model: $\ell_1 \succeq \ell_2$ if $(\ell_1, \ell_2) \to (\ell_1', \ell_2')$ after a sequence of simplifications or cancellations and $\ell_1' \succeq \ell_2'$.

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Length Based Deduction

$\ell_1 \succeq \ell_2$ if $(\ell_1, \ell_2) \to (\ell_1', \ell_2')$ after a pair of simplification or replacements and $\ell_1' \succeq \ell_2'$

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Complexity Aversion

Complexity Aversion

$\succeq$ is *complexity averse* if whenever $\ell_1 S \ell_2$ then $\ell_2 \succeq \ell_1$.

Length Procedural Model

Length Procedural Model

1. Rules:
   - Replacement.
   - Adjacent Simplification.
2. Choice:
   - $\delta_x \succ \delta_y$ if $x > y$.
   - $\succeq$ is complexity averse.
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Proportion Procedural Model

Proportion Procedural Model

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   - $\delta_x \succ \delta_y$ if $x > y$.
   - $\succeq$ is complexity averse.
   - $\succeq$ is mixture continuous
   - $\ell_1 \succeq \ell_2$ if $(\ell_1, \ell_2) \to (\ell_1', \ell_2')$ after a sequence of parallel simplifications or cancellations and $\ell_1' \succeq \ell_2'$

Representation of Complexity Aversion

Theorem 2

Let $\succeq$ be complete and transitive then the following are equivalent:

- $\succeq$ arises out of a length/proportion procedural model
- $\succeq$ is represented by a function V where:

$$V(\ell) = \sum_{x \in supp(\ell)} p_\ell(x)u(x) - C(\ell)$$

$C(\ell)$ is an increasing function of the support size of ℓ.
$C(\ell)$ is the entropy of ℓ multiplied by some constant.

- Axiomatic representations in Puri (2020) and Mononen (2021).

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- Sanity check: the above models are equivalent to EU.

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  - If I have a calculator, evaluating arithmetic equations isn't hard.
  - If my calculator can only add, then equations with division might be harder than multiplication.
- With length based deductions $\Rightarrow$ support size.
- With proportion based deductions $\Rightarrow$ entropy.

Partition Complexity

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- An example:

ℓ_1		ℓ_2	
\$20	20%	\$5.20	20%
\$1.25	20%	\$5.00	20%
\$1.50	20%	\$5.10	20%
\$9.00	20%	\$5.25	20%
\$9.50	20%	\$5.50	20%

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- Same outcome number and entropy, but ℓ_1 is harder to evaluate.

Partition Complexity

Motivation for Model:

- Rules perspective: merging outcomes which are close, cardinally, in values might be easier.
- Complexity perspective: complexity should not only be about "composition" of outcomes, also may depend on cardinal values.

Partition Complexity

Partition Complexity

For each ℓ there is a partition P_ℓ of outcomes in ℓ such that each cell is contained in mutually disjoint intervals and:

$$V_P(\ell) = \sum_{x \in supp(\ell)} p_\ell(x)u(x) - C(|supp(P_\ell)|)$$

Intuition: DM merges, costlessly, outcomes which are close, then evaluates complexity by remaining outcomes.

Procedural Model

- Consider a DM who performs the following:

  1. For each lottery, some simplifications classified as *easy*, others as *hard*.
  2. Always performs the *easy* simplifications first.
  3. Complexity averse towards the number of *hard* simplifications.

Procedural Model

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  1. For each lottery, some simplifications classified as *easy*, others as *hard*.
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- **Result**: The above DM (with some regularity assumption) $\Leftrightarrow$ Partition Complexity Representation.

  details

Numerical Exercise

- Data: CPC-18, aggregate choice probability for 171 choices.
- Cross Validation task: training sample of 30, testing sample of 141.
- Method: CARA utility with Gumbel error.

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Table: Cross Validation Task

	Expected Utility		Entropy Cost		Support Size Cost		Partition Complexity	
	MSE Train	MSE Test	MSE Train	MSE Test	MSE Train	MSE Test	MSE Train	MSE Test
Mean $\times 100$	6.78	6.83	6.77	6.87	6.72	6.84	6.27	6.66
SD $\times 100$	(0.29)	(1.37)	(0.29)	(1.37)	(0.28)	(1.33)	(0.30)	(1.36)

- Paired t-test: Partition complexity has lower error both testing and training ($p < 0.001$) than all three other models.

Thank You!

Partition Complexity Details

Definition

We say a decision-maker uses *fixed simplifications* if for each lottery p she uses a fixed sequence of simplification each time.

Definition

For each lottery, the set of simplification is divided in easy and hard simplifications, a sequence of simplification is *efficient* for this lottery if the easy simplifications occur before hard simplifications.

Parition Complexity Details

Definition

Partition Based Deduction

$\forall \ell_1, \ell_2$, $\ell_1 \succeq \ell_2$ if $(\ell_1, \ell_2) \to (\ell_1', \ell_2')$ is a fixed and efficient sequence which contain n_p, n_q many *hard* simplifications and:

- $n_1 \leq n_2$
- $\ell_1' \succeq \ell_2'$

Partition Complexity Details

The Partition Procedural Model

1. Rules:
   - Adjacent Simplifications.
2. Choice:
   - $\forall x, y \in \mathcal{X}$, $x > y$ implies $\delta_x \succ \delta_y$.
   - Partition Based Deduction

Back